



# Elements of Communication Theory & Signals

**Slot 2 – Baseband Digital Transmission, Bandwidth Limitation and Pulse Shaping**

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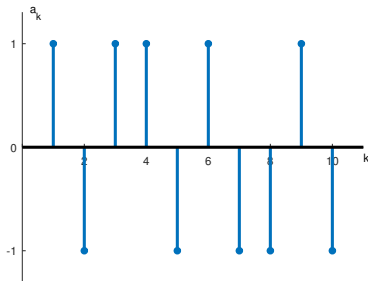
## Definition

- Analog modulation
  - The modulating signal is analog (continuous in time and amplitude)
- Digital modulation
  - The modulating signal is digital (discrete in time and amplitude)

## Note

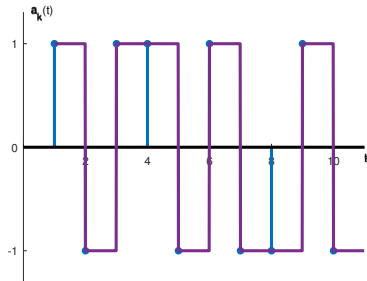
Though "smoothed" (=filtered) digital signals tend to be "analog" signals, the carried information remains coded on discrete states, hence the word "digital" is kept for these signals.

## ■ Ideal binary pulses



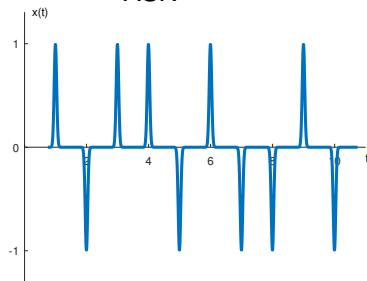
## ■ Binary square pulses

- (baseband) ASK



## ■ Particular shaped pulses

- (baseband) "Filtered" ASK



Recovering the original binary sequence from the received signal requires sampling at the correct instants.

# Why Do Communication Receivers Sample?

Modern receivers ultimately process signals numerically.

Already mentioned: sampling converts  $x(t)$  into  $x[n] = x(nT_s)$ .

Once sampled, operations such as

- filtering;
- synchronization;
- demodulation;
- symbol detection;
- channel decoding

can be implemented numerically.

## The essential question

How fast must we sample so that the discrete-time samples preserve all information contained in the continuous-time waveform?

Represent the sampling instants by an impulse train (also called a Dirac comb):

$$\text{III}_{T_s}(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s). \quad (1)$$

The sampled signal is

$$x_s(t) = x(t)\text{III}_{T_s}(t) = \sum_{n=-\infty}^{+\infty} x(nT_s)\delta(t - nT_s). \quad (2)$$

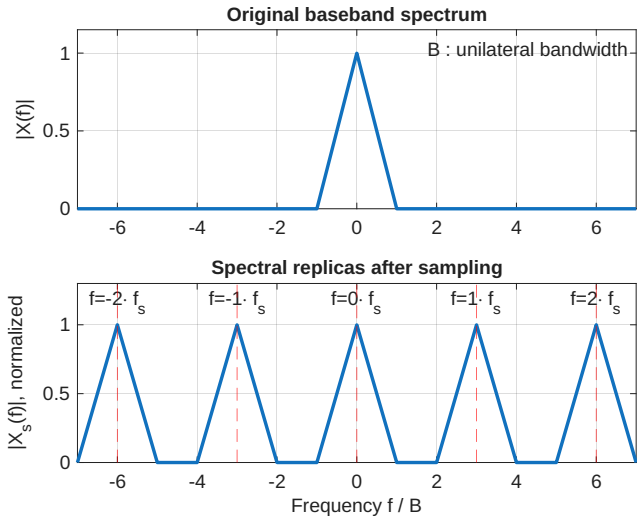
Recall that:

$$\mathcal{F}\{\text{III}_{T_s}(t)\} = \frac{1}{T_s}\text{III}_{f_s}(f) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} \delta(f - kf_s), \quad (3)$$

then:

$$\mathcal{F}\{x_s(t)\} = X_s(f) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X(f - kf_s) \quad (4)$$

# Sampling Creates Spectral Replicas



# Sampling Theorem and Aliasing

Ideal reconstruction is possible when spectral replicas do not overlap.

## Theorem (Sampling Theorem a.k.a. Nyquist–Shannon Sampling Theorem)

*For a real low-pass signal of bandwidth  $B$ , ideal reconstruction from samples is possible if the sampling frequency satisfies*

$$f_s > 2B. \quad (5)$$

If  $f_s < 2B$ , replicas overlap. This is **aliasing**.

## Definition : Nyquist frequency

The Nyquist frequency is defined as half of the sampling frequency:

$$f_{\text{Nyquist}} = \frac{f_s}{2}.$$

(6)

For a real low-pass signal of bandwidth  $B$ , define an oversampling ratio

$$\text{OSR} = \frac{f_s}{2B}. \quad (7)$$

Thus  $\text{OSR} > 1$  provides sampling margin.

We define the symbol rate as:

$$R_{\text{symp}} = \frac{1}{T_{\text{symp}}} \quad (8)$$

where  $T_{\text{symp}}$  is the symbol period of the digital waveform. The **number of samples per symbol** is

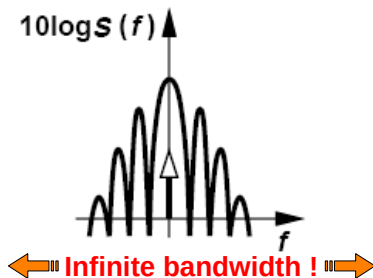
$$N_{\text{spsym}} = \frac{f_s}{R_{\text{symp}}} = f_s T_{\text{symp}}. \quad (9)$$

# Impairments : Channel BW limitation a.k.a. ISI

## Spectrum of (baseband) squared ASK (♣)

- The spectrum of a random binary sequence with equal probabilities of ONEs and ZEROS is given by

$$S(f) = T_b \left( \frac{\sin \pi f T_b}{\pi f T_b} \right)^2 + \frac{1}{2} \delta(f) \quad (10)$$

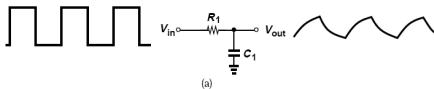
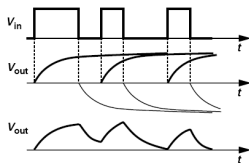
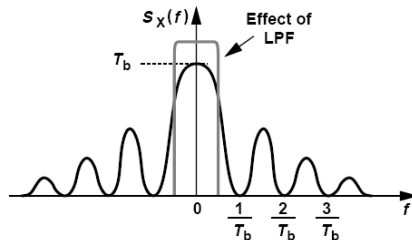


# Impairments : Channel BW limitation a.k.a. ISI

## Intersymbol Interference : definition (♣)

### Channel bandwidth limitation

But channel and electronics are bandlimited systems



- Each bit level is corrupted by decaying tails created by previous bits.
  - **Intersymbol Interference**

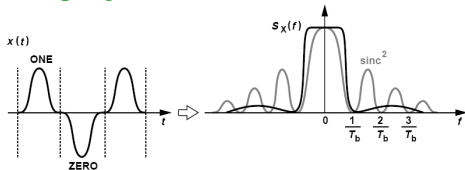
- Low pass filtering creates ISI (except. . . see next slide)

# Impairments : Channel BW limitation a.k.a. ISI

## Pulse Shaping (♣)

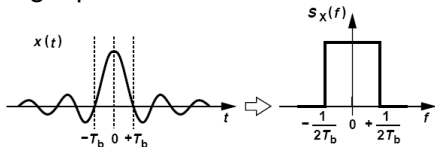
- Special pulse shapes (a.k.a filters) are designed to minimise ISI
  - They reduce the initial bandwidth

### Ex : Sinc

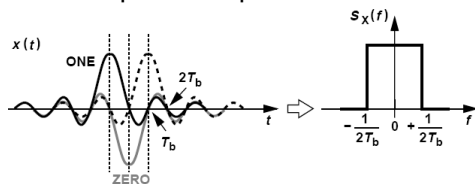


- Baseband pulse is designed to occupy a limited bandwidth.

### Single pulse



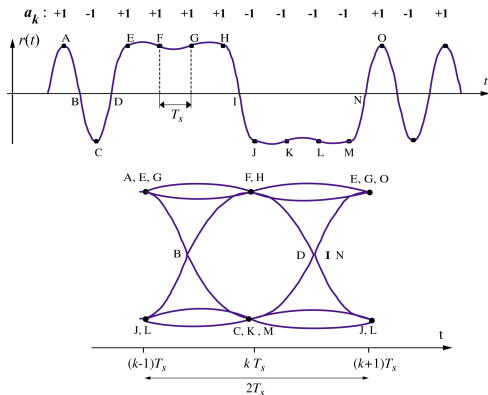
### Random pulse sequence



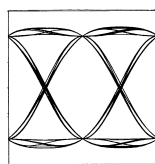
# Impairments : Channel BW limitation a.k.a. ISI

## ISI characterization : eye diagram

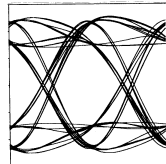
Empirical measure of the quality of the received digital baseband signal



### Binary modulation

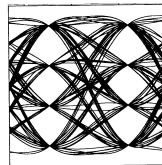


No ISI



With ISI

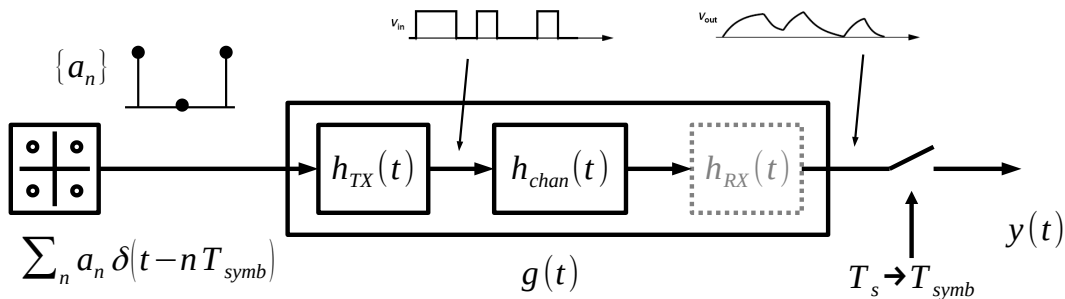
### M-ary modulation



The wider the "eye" opens, the better the signal quality is.

# Impairments : Channel BW limitation a.k.a. ISI

## Nyquist criterion for ISI Cancellation (time domain)



- Effects of ISI could be completely nullified if, at every sampling instant, the response due to all symbols except the current symbol is made equal to zero

# Impairments : Channel BW limitation a.k.a. ISI

## Receiver input and ISI cancellation formula

Output of the overall channel:

$$y(t) = \sum_{n \in \mathbb{Z}} a_n g(t - nT_{\text{symp}}) \quad (11)$$

Let's focus on symbol  $k$ :

$$y(t) = a_k g(t - kT_{\text{symp}}) + \sum_{n \in \mathbb{Z}, n \neq k} a_n g(t - nT_{\text{symp}}) \quad (12)$$

Signal value at the sampling instant  $kT_{\text{symp}}$ :

$$y(t) |_{t=kT_{\text{symp}}} = a_k g(0) + \sum_{n \in \mathbb{Z}, n \neq k} a_n g((k - n)T_{\text{symp}}) \quad (13)$$

- $a_k g(0)$ : magnitude of the expected pulse
- $\sum_{n \in \mathbb{Z}, n \neq k} a_n g((k - n)T_{\text{symp}})$  perturbation term due to ISI

The condition to cancel the ISI term is:

$$\forall a_n \sum_{n \in \mathbb{Z}, n \neq k} a_n g((k - n)T_{\text{symp}}) = 0$$

$$\Rightarrow \forall a_{k-p} \sum_{p \in \mathbb{Z}^*} a_{k-p} g(pT_{\text{symp}}) = 0 \quad (14)$$

## ISI cancellation

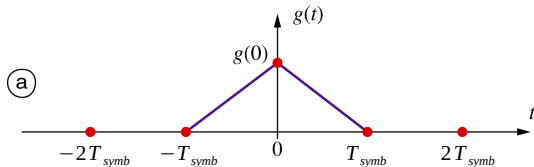
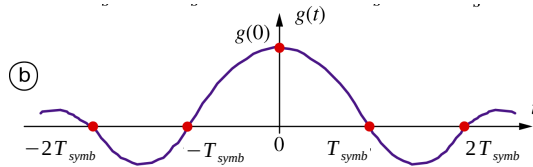
We deduce a *necessary and sufficient condition* for canceling the ISI:

$$\boxed{\begin{cases} \forall p \neq 0, g(pT_{\text{symp}}) = 0 \\ g(0) \neq 0 \end{cases}} \quad (15)$$

## Nyquist criterion for ISI Cancellation (time domain)

$$\begin{cases} \forall p \neq 0, g(pT_{\text{symp}}) = 0 \\ g(0) \neq 0 \end{cases} \quad (15) \Leftrightarrow g(t) \cdot \text{III}_{T_{\text{symp}}} = g(0) \cdot \delta(t) \quad (16)$$

### Examples



# Impairments : Channel BW limitation a.k.a. ISI

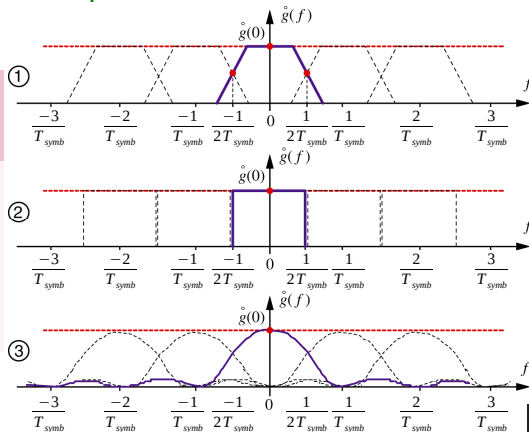
## Nyquist criterion for ISI Cancellation (frequency domain)

### Examples

Applying Fourier transform on Eq. (16) yields:

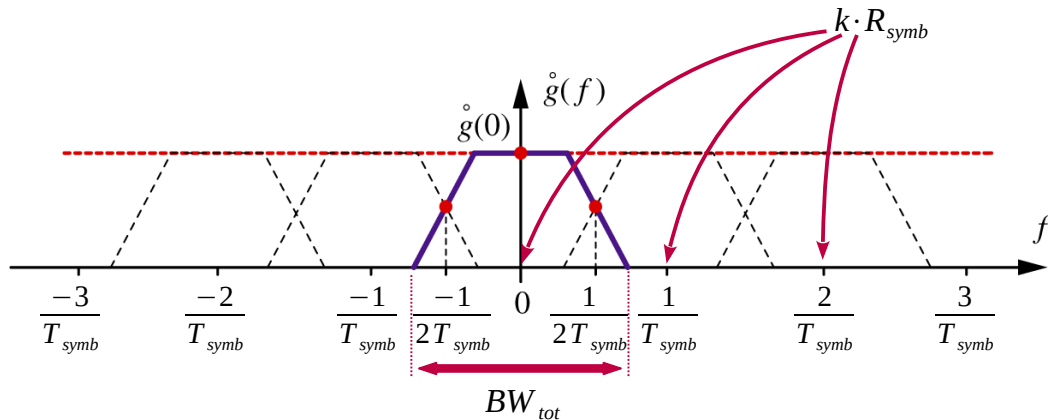
Nyquist criterion for ISI Cancellation (freq domain)

$$\sum_{k \in \mathbb{Z}} \hat{g} \left( f - \frac{k}{T_{\text{symb}}} \right) = T_{\text{symb}} \cdot g(0) \quad (17)$$



# Impairments : Channel BW limitation a.k.a. ISI

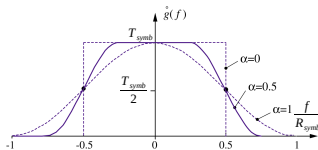
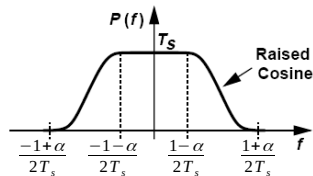
General quantities for a simple case



# Impairments : Channel BW limitation a.k.a. ISI

## Practical pulse shaping filter : Raised Cosine Filter

### ■ Frequency response



$$\hat{g}(f) = \begin{cases} \frac{T_{\text{symb}}}{2} (1 - \sin [\frac{\pi}{2\alpha} (2|f| T_{\text{symb}} - 1)]) , & (1 - \alpha) \frac{R_{\text{symb}}}{2} \leq |f| \leq (1 + \alpha) \frac{R_{\text{symb}}}{2} \\ T_{\text{symb}}, & 0 \leq |f| \leq (1 - \alpha) \frac{R_{\text{symb}}}{2} \\ 0, & \text{elsewhere} \end{cases} \quad (18)$$

- $\alpha$ : roll-off factor, typical values are in the range of  $0.3 \sim 0.5$
- The 3dB bandwidth is the same ! The smaller  $\alpha$ , the steeper the response.

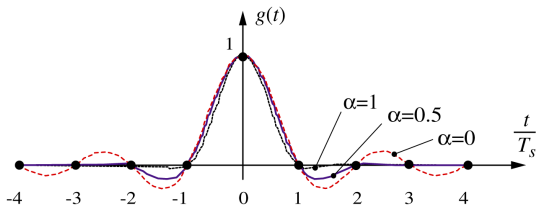
- Theoretical frequency support (*Null-to-null* total bandwidth = **channel usage**):

$$BW_{\text{RaisedCos}} = \frac{1 + \alpha}{T_{\text{symb}}}$$

# Impairments : Channel BW limitation a.k.a. ISI

## Practical pulse shaping filter : Raised Cosine Filter

### ■ Time domain response



$$g(t) = \frac{\sin\left(\frac{\pi t}{T_s}\right)}{\frac{\pi t}{T_s}} \times \frac{\cos\left(\frac{\pi \alpha t}{T_s}\right)}{1 - \left(\frac{2\alpha t}{T_s}\right)^2} \quad (20)$$

- Cancellation instants are the same !
- The smaller  $\alpha$ , the higher the overshoot

|                       | Smaller $\alpha$ | Larger $\alpha$ |
|-----------------------|------------------|-----------------|
| Occupied bandwidth    | smaller          | larger          |
| Frequency transition  | sharper          | smoother        |
| Time-domain ringing   | stronger         | weaker          |
| Implementation margin | lower            | higher          |

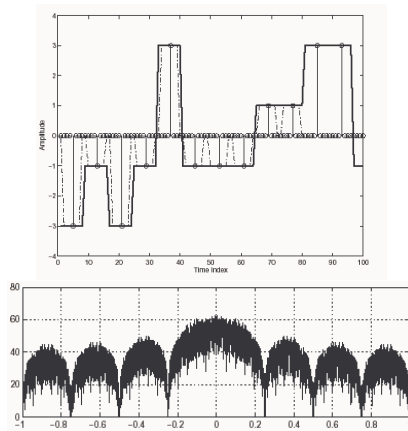
### Central design trade-off

The roll-off factor trades spectral efficiency against practical time-domain localization and filter realizability.

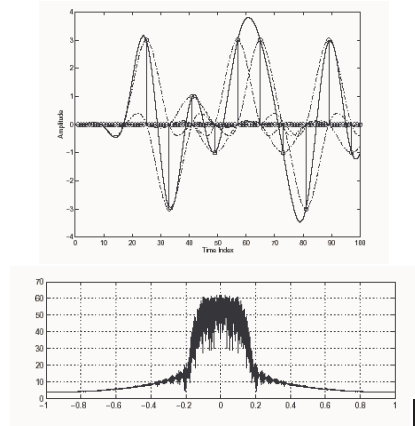
# Impairments : Channel BW limitation a.k.a. ISI

## Signal spectrum after pulse shaping (♦)

### ■ Before pulse-shaping



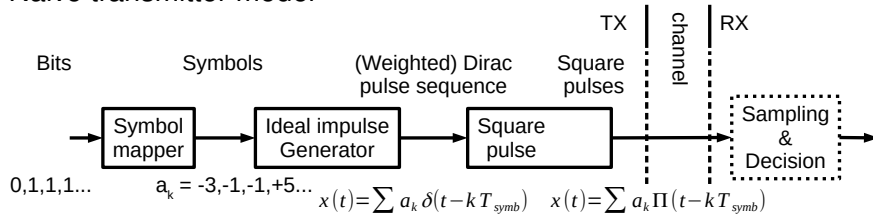
### ■ After pulse-shaping



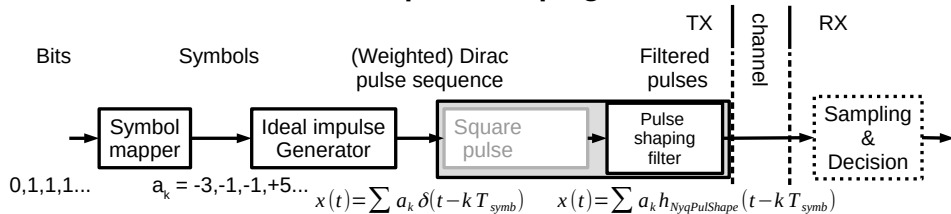
# Impairments : Channel BW limitation a.k.a. ISI

## Conclusion : Transmitter model

### ■ Naive transmitter model



### ■ Now, the transmitter exhibits a **pulse shaping filter**



## Do not confuse two different “Nyquist” results

**Sampling theorem:** prevents spectral aliasing in sampled signals.

**Nyquist zero-ISI criterion:** constrains pulse shapes so symbols do not interfere at the receiver sampling instants.

## Exercise 1 – Intersymbol Interference at the Sampling Instant I

Consider a binary baseband transmission with symbols  $a_k \in \{-1, +1\}$  and symbol period  $T_{\text{symp}}$ . The received waveform, after the overall transmit–channel–receive filtering, is  $y(t) = \sum_{n=-\infty}^{+\infty} a_n g(t - nT_{\text{symp}})$  where  $g(t)$  is the overall pulse response. Suppose that the relevant samples of  $g(t)$  are

$$g(0) = 1, \quad g(\pm T_{\text{symp}}) = 0.20, \quad g(\pm 2T_{\text{symp}}) = 0.05, \quad (21)$$

and all other samples  $g(nT_{\text{symp}})$  are negligible.

## Exercise 1 – Intersymbol Interference at the Sampling Instant II

1. Write  $y(kT_{\text{symb}})$  as the sum of the desired symbol contribution and an ISI term. State the Nyquist zero-ISI condition.

2. For

$$a_{k-2} = a_{k-1} = a_{k+1} = a_{k+2} = +1, \quad a_k = -1, \quad (22)$$

calculate  $y(kT_{\text{symb}})$ .

3. Determine the maximum possible magnitude of the ISI contribution. With a decision threshold at zero, can ISI alone cause an error? Quantify the reduction of the decision margin.

## Exercise 1 – Solution I

At the sampling instant  $t = kT_{\text{symp}}$ ,  $y(kT_{\text{symp}}) = \sum_{n=-\infty}^{+\infty} a_n g[(k-n)T_{\text{symp}}]$ .  
Separating the current symbol  $n = k$ ,

$$y(kT_{\text{symp}}) = a_k g(0) + \sum_{\substack{n=-\infty \\ n \neq k}}^{+\infty} a_n g[(k-n)T_{\text{symp}}] \quad (23)$$

where

$$\underbrace{a_k g(0)}_{\text{desired symbol}} \quad \underbrace{\sum_{n \neq k} a_n g[(k-n)T_{\text{symp}}]}_{\text{ISI}} \quad (24)$$

The Nyquist zero-ISI condition is therefore

$$g(nT_{\text{symp}}) = \begin{cases} g(0), & n = 0, \\ 0, & n \neq 0. \end{cases} \quad (25)$$

**Interpretation:** the pulse may overlap with adjacent pulses between sampling instants; only its values at the decision instants must vanish.

For the specified sequence  $a_k = -1$ , and the four neighboring symbols are  $+1$ . Therefore

$$y(kT_{\text{symb}}) = a_k g(0) + a_{k-1} g(T_{\text{symb}}) + a_{k+1} g(-T_{\text{symb}}) \\ + a_{k-2} g(2T_{\text{symb}}) + a_{k+2} g(-2T_{\text{symb}}).$$

$$y(kT_{\text{symb}}) = (-1)(1) + (+1)(0.20) + (+1)(0.20) \\ + (+1)(0.05) + (+1)(0.05) \\ = -1 + 0.50.$$

$$y(kT_{\text{symb}}) = \boxed{-0.50}.$$

The desired value was

$$a_k g(0) = -1.$$

The neighboring symbols therefore shift the received sample toward the decision threshold.

**Interpretation:** ISI has not yet created an error, but it has reduced the available noise margin.

The maximum magnitude of the ISI occurs when the neighboring symbols add coherently:

$$\begin{aligned} |\text{ISI}|_{\max} &= 2|g(T_{\text{symp}})| + 2|g(2T_{\text{symp}})| \\ &= 2(0.20) + 2(0.05) \\ &= \boxed{0.50}. \end{aligned}$$

The useful symbol contribution has magnitude

$$|a_k g(0)| = 1.$$

Hence ISI alone cannot cross the zero decision threshold because

$$|\text{ISI}|_{\max} < 1. \quad (28)$$

However, the minimum distance from the received sample to the threshold becomes

$$1 - 0.50 = \boxed{0.50}, \quad (29)$$

instead of 1 without ISI.

The decision margin has therefore been reduced by

$$\boxed{50\%}. \quad (30)$$

## Exercise 2 – Raised-Cosine Pulse-Shaping Design I

A baseband communication system transmits symbols at

$$R_{\text{symbol}} = 2 \text{ Msymbol/s.} \quad (31)$$

The pulse-shaping filter has a raised-cosine response with roll-off factor  $\alpha$ .

Recall that  $T_{\text{symbol}} = \frac{1}{R_{\text{symbol}}}$ , and that the raised-cosine spectrum is non-zero for

$$|f| \leq \frac{1 + \alpha}{2 T_{\text{symbol}}}. \quad (32)$$

yielding the total two-sided null-to-null bandwidth

$$B_{\text{NN}} = \frac{1 + \alpha}{T_{\text{symbol}}} = (1 + \alpha) R_{\text{symbol}}. \quad (33)$$

## Exercise 2 – Raised-Cosine Pulse-Shaping Design II

1. For  $\alpha = 0.30$ , determine:
  - the symbol period  $T_{\text{symp}}$ ;
  - the frequency up to which the spectrum is flat;
  - the maximum positive baseband frequency;
  - the total null-to-null bandwidth.
2. The available total baseband bandwidth is now limited to 2.4 MHz. Determine the largest permissible value of  $\alpha$  if  $R_{\text{symp}} = 2$  Msymbol/s must be maintained.
3. If instead  $\alpha = 0.35$  and the available total bandwidth is 3 MHz, determine the maximum allowable symbol rate.

The symbol rate is

$$R_{\text{symp}} = 2 \text{ Msymbol/s.} \quad (34)$$

so

$$T_{\text{symp}} = \frac{1}{R_{\text{symp}}} = \frac{1}{2 \times 10^6} = \boxed{0.5 \mu\text{s}}. \quad (35)$$

For a raised-cosine spectrum, the flat region extends to

$$f_1 = \frac{1 - \alpha}{2 T_{\text{symp}}} \frac{(1 - \alpha) R_{\text{symp}}}{2}. \quad (36)$$

With  $\alpha = 0.30$ ,

$$f_1 = \frac{0.70 \times 2 \text{ Msymbol/s}}{2} = \boxed{0.70 \text{ MHz}}. \quad (37)$$

Therefore the total two-sided null-to-null bandwidth is

$$B_{\text{NN}} = (1 + \alpha)R_{\text{symp}} = \boxed{2.60 \text{ MHz}}. \quad (38)$$

Now, the available bandwidth is  $B_{\text{NN,max}} = 2.4 \text{ MHz}$ , with  $R_{\text{symp}} = 2 \text{ Msymbol/s}$ .  
Therefore

$$(1 + \alpha)R_{\text{symp}} \leq B_{\text{NN,max}},$$
$$\alpha \leq \left( \frac{B_{\text{NN,max}}}{R_{\text{symp}}} - 1 \right) = \left( \frac{2.4 \text{ MHz}}{2 \text{ Msymbol/s}} - 1 \right) = 0.2.$$

Hence

$$\boxed{\alpha_{\text{max}} = 0.20}. \quad (39)$$

Now let  $\alpha = 0.35$  and  $B_{\text{NN,max}} = 3 \text{ MHz}$ .

The maximum symbol rate is

$$R_{\text{symp,max}} = \frac{B_{\text{NN,max}}}{1 + \alpha} \frac{3 \text{ MHz}}{1.35} \approx \boxed{2.22 \text{ Msymbol/s}}. \quad (40)$$

The corresponding minimum symbol period is

$$T_{\text{symp,min}} = \frac{1}{R_{\text{symp,max}}} \approx \boxed{0.45 \text{ } \mu\text{s}}. \quad (41)$$